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Finite groups with weakly-supplemented subgroups of prime orders

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Abstract
A subgroup H of a finite group G is weakly-supplemented in G if there exists a proper subgroup K of G such that G=HK. In the paper, one interesting result with weakly-supplemented minimal subgroups of G is obtained. Some known results are generalized.

Keywords: Finite group, weakly-supplemented subgroups, supersolvable groups

1. Introduction

It is well known that a subgroup H of a finite group G is complemented in G if there exists a subgroup K of G such that G=HK and $H \cap K = 1$. Such a subgroup K of G is called a complement to H in G. It is clear that the existence of complements for certain subgroups of a finite group gives a lot of useful information about its structure, such as in [1, 2, 4-7] etc. For instance, P. Hall in [4] proved that a finite group is solvable if and only if every Sylow subgroup of G is complemented. Arad and Ward in [1] proved that a finite group is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented. In particular, Hall in [5] proved that a finite G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G. Ballester-Bolinches and Guo in [2] shown that: Let P be a Sylow p-subgoups of G, where G is a finite group and p is the smallest prime divisor of |G|. If every maximal subgroup of P is complemented in G, then G is p-nilpotent. Qiao, Isaacs and Li in [7] extended the above result and proved that: Let P be a Sylow p-subgoups of G, where G is a finite group and p is the smallest prime divisor of |G|. Let d be a power of p with 1<d<|P|, and assume that every subgroup of P having order d is complemented in G. Suppose also that N_{G}(P) is p-nilpotent. Then G is p-nilpotent. In a recent paper, Kong and Liu in [6] studied finite groups for which every minimal subgroup is weakly-supplemented. A subgroup H of G is weakly-supplemented in G if there exists a proper subgroup K of G such that G=HK. They proved that every minimal subgroup of G is weakly-supplemented in G if and only if G is a supersolvable group and all Sylow subgroups of G are elementary abelian.

In this note, we further investigate the influence of weakly-supplemented subgroups on the structure of finite groups along the above direction. Our main result is the following:

Theorem A. Let G be a finite group, $p \in \pi(G)$. Every subgroup of order p is weakly-supplemented in a p-solvable group G if and only if G is a p-supersolvable group with elementary abelian Sylow p- subgroups.

All groups considered in this paper are finite groups. Our notation and terminology are standard. The reader may refer to ref. [3].

2. Preliminary results

In this section, we give one lemma which are useful for our main results.

Lemma 2.1 ([6]) Let G be a group. $p \in \pi(G)$, Suppose that every subgroup of G of order p is weakly-supplemented in G. Then:

i. If H is a subgroup of G, then every subgroup of H of order p is weakly-supplemented in H; in particular, a Sylow p-subgroup of G is elementary abelian and weakly-supplemented in G.

ii. If N is a normal subgroup of G and G is p-solvable, then every subgroup of G/N of order p is weakly-supplemented in G/N.

3. Proof of the Main Theorem

Proof of Theorem A. By Lemma 2.1(i), every Sylow p-subgroup of G is elementary abelian and weakly-supplemented in G. Let us verify that G is p-supersolvable. We use induction on the order of G. By Lemma 2.1(ii), the conditions of the theorem hold for all quotient groups, so we may assume $O_{p'}(G)=\Phi(G)=1$, $N=F(G)=O_p(G)=C_G(F(G))$, and N is a minimal normal subgroup of G. It follows that G has a maximal subgroup M with G=[N]M. Let X be a subgroup of N of prime order p. By the hypotheses of the theorem, there is a proper subgroup Y of G such that G=XY. The subgroup $N_1=N\cap Y$ is normal in Y and centralized by X. Hence N_1 is normal in G. But N is a minimal normal subgroup of G. Thus $N_1=1$, N has prime order p, so G is p-supersolvable. Thus necessity is proved.

Now we prove sufficiency. Let G be a p-supersolvable group with elementary abelian Sylow p- subgroups . Fix an arbitrary subgroup A of prime order p. Suppose that there exists a normal subgroup $N\neq 1$ such that A is not contained in N. Then $A\cap N=1$ and by induction, the subgroup AN/N is weakly-supplemented in G/N. Suppose that B/N is weakly-supplement for AN/N in G/N. Then G/N=(AN/N)B/N, $AN/N\cap B/N=N/N$, G=(AN)B=AB, and $A\cap B$ =1 since $|G:B|=p$. So we may assume that every normal subgroup of the group G contians A. Since every minimal normal subgroup of a p-supersolvable group either has order p or is a p' - subgroup, it follows that A is the only minimal normal subgroup of G. But A is weakly-supplemented in the Sylow p-subgroup since the latter is elementary abelian. The theorem of Gaschutz of [3, Theorem I.17.4] implies that the subgroup A is weakly-supplemented in G. Thus our proof is complete now.

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Rreferences

1. Arad Z, Ward MB. New criteria for the solvability of finite groups, J. Algebra 1982;77:234-246.
2. Ballester-Bolinches A, Guo X. On complemented subgroups of finite groups, Arch. Math., 1999;72:161-166.
3. Huppert B, Endliche Gruppen I, Springer-Verlag, Berlin-Heidelberg-New York, 1967
4. Hall P. A characteristic property of soluble groups, J. London Math. Soc. 1937;12:188-
5. Hall P. Complemented groups, J. London Math. Soc 1937;12:201-204.
6. Kong Q, Liu Q. The influence of weakly-supplemented subgroups on the structure of finite groups, Czechoslovak Math. J. 2014;64(139):173-182.
7. Qiao S, Isaacs IM, Li Y. complemented subgroups and p-nilpotence in finite groups, Arch. Math., 2012;98:403-411.